Why do We Care about Power Series?

Febuary 19, 2025

One motivation is for application in numeric differentiation, and finding approximate solutions to differential equations:

Overview:

1.) Motivating Problem
2.) Taylor's Theorem
3.) Foward Difference Method
4.) Backward Difference Method
5.) Polynomial approximation/interpolation.

1.) Motivating Problem

We are given a function $g:[a,b]\to\mathbb{R}$, and ask to find a function $f:[a,b]\to\mathbb{R}$

$$f' + f^2 = g$$

where $f(a)=c$ initial condition.

This is an example of a first order, non-linear, ordinary differential equation.

We don't know what values f take, only $f(a)=c$.

Idea: ① Find $\{x_0, \dots, x_n\} \subseteq [a,b]$, and $\{f_0, \dots, f_n\} \subseteq \mathbb{R}$ using methods below,

where $f(x_i) \approx f_i$.

② Use polynomial interpolation to to find a polynomial that approximates f.

2.) Taylor's Theorem

Thm (Taylor's Expansion)

If $f : [a,b] \to \mathbb{R}$ a $n+1$ times differentiable function, $x_0 \in [a,b]$ fixed, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + R_n(x) \quad , \text{ where}$$

the error term is

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\alpha(x)) \cdot (x-x_0)^{n+1}$$

where $\alpha = \alpha(x) \in [a,b]$. ($\alpha$ depend on x)

3.) Foward Difference Method

We make the assumption that our solution f is at least 4 times differentiable.

Then by Taylor's expansion at $x_0 \in [a,b]$

$$(*) \quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{6} f'''(x_0)(x-x_0)^3 + R_3(x), \quad \forall x \in [a,b]$$

where $R_3(x) = \dfrac{f^{(4)}(\alpha(x))}{4!}(x-x_0)^4$, some

$\alpha = \alpha(x) \in [a,b]$.

- Now fix $0<h<1$, and replace
  - x_0 with x
  - x with $x+h$

Difference between forward & backward.

Then $(*)$ turns into

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x) h^2 + \frac{1}{6} f'''(x) h^3 + \frac{f^{(4)}(\alpha)}{4!} h^4$$

$$= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + O(h^4)$$

Now solve for $f'(x)$, $(h>0)$

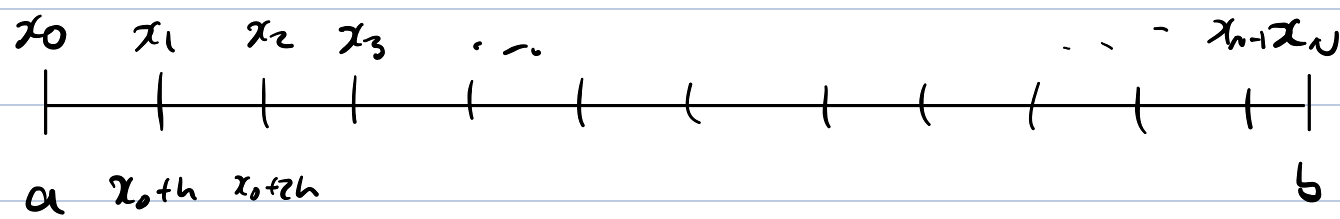
$$f'(x) = \frac{f(x+h)-f(x)}{h} \underbrace{- \frac{1}{2} f''(x)h - \frac{1}{6} f'''(x)h^2 + O(h^3)}_{\circledast}$$

Since we want $0<h<1$, and we want $h \to 0^+$, h is the dominati term in $\circledast$. So it becomes $O(h)$.

Hence

$$\#\# \quad f'(x) = \frac{f(x+h)-f(x)}{h} - O(h).$$

• Here we start discritizing:



Divide $[a,b]$ into a partision of intervals of length h, s.t $x_k = x_0 + k \cdot h$.

Observe that $x_k + h = x_0 + kh + h = x_{k+1}$.

Then consider

$$f'(x_j) = \frac{f(x_j + h) - f(x_j)}{h} - O(h)$$

$$= \frac{f(x_{j+1}) - f(x_j)}{h} - O(h)$$

forward

$$\approx \frac{f_{j+1} - f_j}{h}.$$

Remember that we don't know what $f(x_j)$ is. Here f_j's just a number to use in iteration.

But we do know what $f(x_0) =: f_1$.

Recall we want to solve

$$f'(x_j) + f(x_j)^2 = g(x_j)$$

Algorithm

Step 1: Solve f_1 in:

$$\frac{f_1 - f_0}{h} + f_0^2 = g(x_0)$$

$$\Rightarrow f_1 - f_0 + h f_0^2 = g(x_0)$$

$$\Rightarrow f_1 = \overbrace{g(x_0) + f_0 + h f_0^2}^{\text{This we can compute!}}$$

Step 2: Solve f_2 in:

$$\frac{f_2 - f_1}{h} + f_1^2 = g(x_1)$$

$$\Rightarrow f_2 = \underbrace{g(x_1) + f_1 + h f_1^2}_{\text{we can compute this.}}$$

Continue recursively computing until we are left with $n+1$ distinct points, $\{x_0, \ldots, x_n\}$ and $\{f_0, \ldots, f_n\}$

Now apply polynomial interpolation to find a polynomial s.t

$$p(x_i) = f_i .$$

this is our approximate solution to $f' + f^2 = g$.

Comments:

- Only stable for h very small. i.e conditionally stable.
- Backward difference method gets used more, back its unconditionally stable.

4.) Backward Difference Method

We make the assumption that our solution f is at least 4 times differentiable.

Then by Taylor's expansion at $x_0 \in [a,b]$

$$(*) \quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{6} f'''(x_0)(x-x_0)^3 + R_3(x), \quad \forall x \in [a,b]$$

where $R_3(x) = \frac{f^{(4)}(\alpha(x))}{4!}(x-x_0)^4$, some

$\alpha = \alpha(x) \in [a,b]$.

- Now fix $0<h<1$, and replace
  - x_0 with x
  - x with $x-h$

- Then using Taylor's expansion,

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + O(h^4)$$

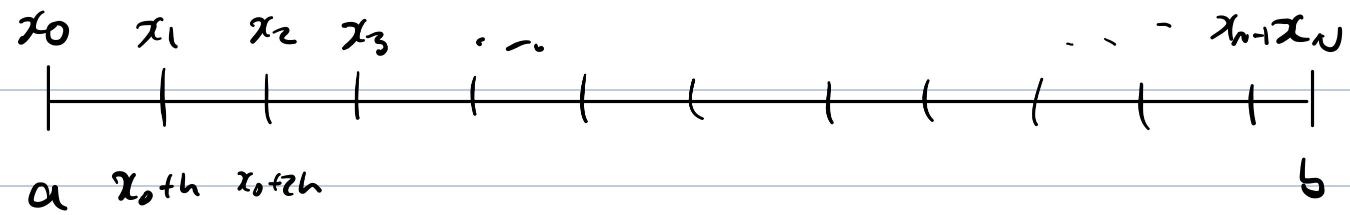
- Solve for $f'(x)$, just like before: $(0<h)$

$$f'(x) = \frac{f(x-h) - f(x)}{h} - \frac{f''(x)}{2} h - \frac{f'''(x)}{6} h^2 + O(h^3)$$

$$= \frac{f(x-h) - f(x)}{h} - O(h)$$

Again discritize, and take a mesh

- Here we start discritizing:



Divide $[a,b]$ into a partision of intervals of length h, s.t

$x_k = x_0 + k \cdot h$

Observe that $x_k - h = x_0 + kh - h = x_{k-1}$.

Thus

$$f'(x_j) = \frac{f(x_j) - f(x_j - h)}{h} + O(h)$$

$$= \frac{f(x_j) - f(x_{j-1})}{h} + O(h)$$

This's why its backward.

$$\approx \frac{f_j - f_{j-1}}{h} + O(h)$$

- Now let's find an approximate solution to

$$f'(x_j) + f^2(x_j) = g(x_j),$$

where $f(a) = c$, and the function g is given.

Algorithm

Step 1: Solve f_1 in

Non-linear since variables are squared

$$\frac{f_1 - f_0}{h} + f_1^2 = g(x_0)$$

- Here we need to solve a non-linear equation, as compared to the forward difference method.

- Indeed we need to solve

$$h_1(x) = 0 \quad \text{where}$$

$$h_1(x) := \frac{x - f_0}{h} + x^2 - g(x_0).$$

- That is apply a fixed point method such as newton

$$h_1(x) + x = x$$

- So find a fixed point for $\tilde{h}(x) := \dfrac{x - f_0}{h} + x^2 + x - g(x_0)$

• The solution to this fixed point problem we call f_1

Step 2: Solve f_2 in

$$\frac{f_2 - f_1}{h} + f_2^2 = g(x_1)$$

- Here we need to solve a non-linear equation, as compared to the forward difference method.

- Again solve the fixed point problem

$h_2(x) = 0$ when

$$h_2(x) := \frac{x - f_1}{h} + x^2 - g(x_1)$$

Continue recursively computing until we are left with $n+1$ distinct points, $\{x_0, \dots, x_n\}$ and $\{f_0, \dots, f_n\}$

Now apply polynomial interpolation to find a polynomial s.t

$p(x_i) = f_i$.

this is our approximate solution to $f' + f^2 = g$.

5.) Polynomial approximation/interpolation.

Problem: Given distinct numbers $n+1$ $x_0, x_1, \ldots, x_n \in [a,b]$, and numbers $y_0, \ldots, \ldots, y_n$, can we find a interpolating polynomial p of degree at most n s.t

$$p(x_i) = y_i \quad \text{for all } i = 0, 1, \ldots, n.$$

- We are thinking there is some function $f : [a,b] \to \mathbb{R}$ in the background that we don't know, but we know what it does on $x_0, \ldots, x_n$ and it sends it to $f(x_i) = y_i$.
- We want to approximate this f.

Can we find such a polynomial?
If so, how?

Different Types of Polynomials to use.

T.B.C.